III-2-4. Normal distribution

III-2-4-1. Deriving normal distribution from binominal distribution

Binominal distribution is distribution of ratio, and the plot is discontinuous. The shape becomes smoother with increase of n. if $\mathrm{n} \to \infty$ the shape will be completelly smooth. n. There is 2 methods of increase of n. One is increasing of only n to $\infty$with fixing p at definite value. This is normal distribution. We already try this in previous chapters. Those results show that the shape becomes symmetric and sharp. The purpose of making normal distribution is application of stochastic discussion to continuous value data such as body length and weight to compare the difference among several sample populations. The other is increasing of n with fixing mean (np) at definite value. Probability (p) necessarily decrease with increase of n and the shape becomes asymmetric. This is Poisson distribution. Poisson distribution is possibility distribution of rarely happening phenomenon. In the field of fisheries science, Poisson distribution is used appearance of rare species in counting frame.

We use Taylor expansion to make normal distribution from binominal distribution. Taylor expansion is a method for simplification of complicated function to polynomial formula using Taylor series. Taylor series is a series of derivative function by sequential derivation. Generally, Taylor expansion is expressed as follow

$$f(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

When n is finite number,

$$f(x) \fallingdotseq \sum_{i=1}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i$$

Here,

$$f^{(i)}; i \text{ order of derivative.}$$

For readers who need more detailed information of Taylor expansion, the author wrote explanation of Taylor analysis in III-3-1.

We already know

$$\frac{d(\log W(x))}{dx} = \log\frac{(n-x)p}{x(1-p)} = -\log x + \log(n-x) + \log p - \log(1-p)$$

$$\frac{d(\log W(x))}{dx} = 0$$

$$\text{at } x = np = \mu$$

So we will do Taylor expansion around μ

Second order derivative is

$$\frac{d^2(\log W(x))}{dx^2} = \{-\log x + \log(n-x) + \log p - \log(1-p)\}'$$

$$= -\frac{1}{x} - \frac{1}{n-x}$$

$$= -\frac{1}{np} - \frac{1}{n-np}$$

$$= -\frac{1}{n}(\frac{1}{p} + \frac{1}{1-p})$$

$$= -\frac{1}{np(1-p)}$$

$$= -\frac{1}{\sigma^2}$$

$$\because np(1-p) = \sigma^2$$

We neglect derivatives higher than third order derivatives.

$$\log \mathrm{W}(x) = \log(\mathrm{n}!) - \log(x!) - \log(\mathrm{n}-x)! + \mathrm{k}\log(\mathrm{p}) + (\mathrm{n}-x)\log(\mathrm{q})$$

$$\fallingdotseq \log \mathrm{W}(\mu) + \frac{(\log(\mu))'}{1!}(x-\mu) + \frac{(\log(\mu))''}{2!}(x-\mu)^2$$

$$= \log W(\mu) + \frac{(\log(x))''}{2}(x-\mu)^2$$

$$= \log W(\mu) + \frac{-\frac{1}{\sigma^2}}{2}(x-\mu)^2$$

Here

$$\log_e e = 1$$

Caution

e is base of natural logarithm. Mathematically, name of e is Napier's constant. In previous part of this text, $\log A$ is used without any explanatory remarks in the meaning $\log_e A$ following habit that when $\log$ is used without base the symbol means $\log_e$. In following exlanation, the author will use $\log_e$ for perspicuous understanding.

Necessary background information for following explanation is

$$\frac{d\log x}{dx} = \frac{1}{x}, \quad \frac{de^x}{dx} = e^x$$

Please refer III-3-2 (Napier's constant), if reader did not have background information or if reader needs detailed explanation of Napier's constant.

$$\log W(x) \fallingdotseq \log W(\mu) + \frac{-1}{2\sigma^2}(x-\mu)^2$$

$$= \log_e W(\mu) + \frac{-1}{2\sigma^2}(x-\mu)^2 \log_e e$$

$$= \log_e W(\mu) + \log_e e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$$= \log_e W(\mu) + \log_e e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$$= \log_e W(\mu) e^{\frac{-1}{2\sigma^2}(x-\mu)^2} = \log_e W(\mu) e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

When we remove logarithm symbol both sides,

$$\mathrm{W}(x) = \mathrm{W}(\mu) e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

When $x = \mu$,

$$e^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} = e^0 = 1$$

We could confirm

$$\mathrm{W}(\mu) = \mathrm{W}(\mu)$$

Meaning of $\mathrm{W}(\mu)$ is the height of the peak of the distribution.
Distance from mean is $x - \mu$
Distance from mean expressed using σ as unit is $\frac{x-\mu}{\sigma}$

$\left(\frac{x-\mu}{\sigma}\right)^2$ is symmetric, ($x = \mu$ is axis of symetry.)

So μ is mean, mode value and median value in normal distribution. We can understand those natures of normal distribution from the formula, though the form of equation is different from general formula which we know as normal distribution.
At least, we want to express $\mathrm{W}(\mu)$ by μ and σ
We calculate $\mathrm{W}(\mu)$ giving constraint condition. Most easily conceivable condition is that sum of $\mathrm{W}(x)$ should be 1, because $\mathrm{W}(x)$ is probability and total sum of probability should be 1. This means integration of $\mathrm{W}(x)$ from $-\infty$ to ∞ is 1.
We can obtain the formula by solving following equation.

$$\int_{-\infty}^{\infty} W(x) e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

For descriptive purpose, $W(x) = A$

$$\int_{-\infty}^{\infty} W(x) e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} A\, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \mathrm{A}\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

We can simplify the formula by

$$\mathrm{X} = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2 = X^2$$

$$\frac{dX}{dx} = \frac{1}{\sqrt{2}\sigma}$$

$$dx = \sqrt{2}\sigma d\mathrm{X}$$

$$\mathrm{A}\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \mathrm{A}\int_{-\infty}^{\infty} e^{-X^2}\sqrt{2}\sigma dX$$

$$= \sqrt{2}\sigma A \int_{-\infty}^{\infty} e^{-X^2} dX$$

Conclusively, we need to calculate following formula.

$$\int_{-\infty}^{\infty} e^{-X^2} dX$$

Before long explanation, the author gives following answer at first.

$$\int_{-\infty}^{\infty} e^{-X^2} dX = \sqrt{\pi}$$

Proof of this equation is as follow.

We know the formula is symmetry.

When

$$I = \int_{-\infty}^{\infty} e^{-x^2} dxdx$$

then

$$\frac{I}{2} = \int_{0}^{\infty} e^{-x^2} dx$$

$$\frac{I^2}{4} = \int_{0}^{\infty} e^{-x^2} dx \times \int_{0}^{\infty} e^{-x^2} dx$$

Here. we consider integration of first definite integration and second definite integration separately,

$$\frac{I^2}{4} = \int_{0}^{\infty} e^{-x^2} dx \times \int_{0}^{\infty} e^{-y^2} dy$$

This is multiplication of definite integrations, In the case x and y are independent each other, multiplication of definite integration is same as multiple integral.

$$\int_0^{\infty} e^{-x^2} dx \times \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty}\int_0^{\infty} e^{-x^2} \times e^{-y^2} dxdy$$

This transformation is not only for simplifying but also for making 3-dimensional probability distribution. Produced 3-dimansional solid is solid of revolution, and we can calculate the cubic volume by integration of a plain with rotation angle. The author wrote the detailed process of the change of coordinate from rectangular coordinates to polar coordinate in III-3-4, III-3-5 (Coordinate conversion and multiple integration) with figures for visualization of the process.

Here, the author will show the process of transformation of formula

$$\int_0^{\infty} e^{-x^2} dx \times \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty}\int_0^{\infty} e^{-x^2} \times e^{-y^2} dxdy$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dxdy$$

Transformation to polar coordinate (See III-3-4. Polar coordinate)

$$x = \mathrm{r}\cos\theta$$

$$y = \mathrm{r}\sin\theta$$

$$\frac{dx}{dr} = \cos\theta$$

$$\frac{dy}{d\theta} = \mathrm{r}\cos\theta$$

$$x^2 + y^2 = r^2$$

$$\int_0^{\infty}\int_0^{\infty} e^{-x^2} \times e^{-y^2} dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\infty} re^{-r^2}\cos^2\theta\, drd\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$

$$\because \mathrm{s} = r^2$$

$$\frac{ds}{dr} = 2r$$

$$dr = \frac{ds}{2r}$$

$$\int_0^{\infty} re^{-r^2} dr = \int_0^{\infty} re^{-s}\frac{ds}{2r} = \int_0^{\infty} e^{-s}ds = -\frac{1}{2}[e^{-s}]_0^{\infty} = -\frac{1}{2}(0-1) = \frac{1}{2}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 2\theta + 1}{2}d\theta = \frac{1}{4}\int_0^{\pi}\frac{\cos\omega}{2}d\omega + \frac{1}{4}\int_0^{\pi}d\omega = \frac{1}{8}[\sin\omega]_0^{\pi} + \frac{1}{4}[\omega]_0^{\pi} = \frac{\pi}{4}$$

Calculation of Jacobian (See III-3-3 Jacobian)

$$\frac{dx}{dr} = \cos\theta$$

$$\frac{dy}{dr} = \sin\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta$$

$$\frac{dy}{d\theta} = r\cos\theta$$

$$J_\varphi = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\left|J_\varphi\right| == \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\because \cos^2\theta + \sin^2\theta = 1$$

$$\frac{I^2}{4} = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\left|J_\varphi\right| drd\theta = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,rdrd\theta$$

About $\int_0^\infty e^{r^2}\,rdr$

$$r^2 = s$$

$$2\mathrm{r} = \frac{ds}{dr}$$

$$rdr = \frac{1}{2}ds$$

$$\int_0^\infty e^{-r^2}\,rdr = \int_0^\infty e^{-s}\frac{1}{2}ds$$

$$= \frac{1}{2}\int_0^\infty e^{-s}\,ds$$

$$= \frac{1}{2}[-e^{-s}]_0^\infty$$

$$= \frac{1}{2}\left\{-\frac{1}{e^\infty} - \left(-\frac{1}{e^0}\right)\right\}$$

$$= \frac{1}{2}\{0 - (-1)\}$$

$$= \frac{1}{2}$$

$$\frac{I^2}{4} = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-s} \frac{1}{2} ds\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2} [\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

Conclusively,

$$\frac{I^2}{4} = \frac{\pi}{4}$$

$$I^2 = \pi$$

$$\mathrm{I} = \sqrt{\pi}$$

From limiting condition

$$\int_{-\infty}^{\infty} A\, e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$$

$$\mathrm{A} \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sqrt{2}\sigma A \int_{-\infty}^{\infty} e^{-X^2}\, dX$$

$$= \sqrt{2}\sigma A I$$

$$= \sqrt{2}\sigma\sqrt{\pi}A$$

$$= \sqrt{2\pi}\sigma \mathrm{A}$$

$$\sqrt{2\pi}\sigma A = 1$$

$$\mathrm{A} = \frac{1}{\sqrt{2\pi}\sigma}$$

$$\mathrm{W}(x) = \mathrm{W}(\mu) e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$$\mathrm{W}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

$$\mathrm{W}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

This is generally used formula of normal distribution.

$\mathrm{N}(\mu, \sigma)$ means normal distribution of which mean and variance are μ and σ^2.

When x follows $\mathrm{N}(\mu, \sigma)$, possibility of x, $\mathrm{P}(x)$ is

$$\mathrm{P}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Formula 17