III-2-7. F distribution

F distribution is probability distribution of ration of two variance. T distribution is combination of normal distribution and χ^2 distribution. F distribution is product of 2 χ^2 distributions. The formulization process is basically similar as the process used in normal distribution, χ^2distribution and t distribution.

There are two variances(z_1, z_2) which distribute in χ^2 distribution. The variance is $\frac{z_1}{n_1}$ and $\frac{z_2}{n_2}$. (n is degree of freedom, not number of data). We consider the ratio of two variances

$$f = \frac{\frac{z_1}{n_1}}{\frac{z_2}{n_2}}$$

$$\mathrm{P}(z_1) = \frac{z_1^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)} e^{\frac{-z_1}{2}}$$

$$\mathrm{P}(z_2) = \frac{z_2^{\frac{n_2}{2}-1}}{2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} e^{\frac{-z_2}{2}}$$

$$\mathrm{F}(f) = \mathrm{P}(z_1)\mathrm{P}(z_2)$$

We have to consider orthogonal plane of z_1 and z_2.

See III-3-3. Jacobian, III-3-4. Polar coordinate, III-3-5. Multiple integral

$$f = \frac{z_1}{n_1}\frac{n_2}{z_2}$$

$$z_1 n_2 = fz$$

$$z_2 n_1 = z$$

$$0 \le \mathrm{z}_1 \le \infty, 0 \le z_2 \le \infty, 0 \le \mathrm{f} \le \infty, 0 \le \mathrm{z} \le \infty$$

$$\int_0^\infty \int_0^\infty \mathrm{P(z)}dz\,df = 1$$

$$\frac{dz_1}{df} = \frac{\mathrm{z}}{n_2}$$

$$\frac{dz_1}{dz} = \frac{f}{n_2}$$

$$\frac{dz_2}{df} = 0$$

$$\frac{dz_2}{dz} = \frac{1}{n_1}$$

$$J(z_1, z_2/f, z) = \begin{bmatrix} \frac{dz_1}{df} & \frac{dz_1}{dz} \\ \frac{dz_2}{df} & \frac{dz_2}{dz} \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{z}}{n_2} & \frac{\mathrm{f}}{n_2} \\ 0 & \frac{1}{n_1} \end{bmatrix} = \frac{z}{n_1 n_2}$$

$$\int_0^\infty \int_0^\infty \mathrm{P(z)}dz\,\mathrm{d}f = \int_0^\infty \int_0^\infty \mathrm{P}(z_1)\mathrm{P}(z_2)J(z_1, z_2/f, z)dz\,\mathrm{d}f = 1$$

$$\mathrm{F}(f) = \frac{d\left(\int_0^f \mathrm{F}(f)df\right)}{df}$$

$$\mathrm{F}(f) = \int_0^\infty \frac{{z_1}^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)} e^{\frac{-z_1}{2}} \frac{{z_2}^{\frac{n_2}{2}-1}}{2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} e^{\frac{-z_2}{2}} \frac{z}{n_1 n_2} dz$$

$$= \int_0^\infty \frac{\left(\frac{fz}{n_2}\right)^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right)} e^{\frac{-\left(\frac{fz}{n_2}\right)}{2}} \frac{\left(\frac{z}{n_1}\right)^{\frac{n_2}{2}-1}}{2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} e^{\frac{-\left(\frac{z}{n_1}\right)}{2}} \frac{z}{n_1 n_2} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1}{2}}\Gamma\left(\frac{n_1}{2}\right) 2^{\frac{n_2}{2}}\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \left(\frac{z}{n_2}\right)^{\frac{n_1}{2}-1} \left(\frac{z}{n_1}\right)^{\frac{n_2}{2}-1} e^{\frac{-\left(\frac{fz}{n_2}\right)}{2}} e^{\frac{-\left(\frac{z}{n_1}\right)}{2}} \frac{z}{n_1 n_2} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \left(\frac{z}{n_2}\right)^{\frac{n_1}{2}-1} \frac{1}{n_2} \left(\frac{z}{n_1}\right)^{\frac{n_2}{2}-1} \frac{1}{n_1} z e^{\frac{-\left(\frac{fz}{n_2}+\frac{z}{n_1}\right)}{2}} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}}\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \frac{z^{\frac{n_1}{2}-1}}{{n_2}^{\frac{n_1}{2}}} \frac{z^{\frac{n_2}{2}-1}}{{n_1}^{\frac{n_2}{2}}} z e^{\frac{-\left(\frac{fz}{n_2}+\frac{z}{n_1}\right)}{2}} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} {n_2}^{\frac{n_1}{2}} {n_1}^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty z^{\frac{n_1+n_2}{2}-2} z e^{\frac{-\left(\frac{fz}{n_2}+\frac{z}{n_1}\right)}{2}} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} {n_2}^{\frac{n_1}{2}} {n_1}^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty z^{\frac{n_1+n_2}{2}-1} e^{-\left(\frac{f}{n_2}+\frac{1}{n_1}\right)\frac{z}{2}} dz$$

$$\left(\frac{f}{n_2} + \frac{1}{n_1}\right)\frac{z}{2} = \mathrm{w}$$

$$z = \frac{2}{\left(\frac{f}{n_2}+\frac{1}{n_1}\right)} \mathrm{w}$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty z^{\frac{n_1+n_2}{2}-1} e^{-\left(\frac{f}{n_2}+\frac{1}{n_1}\right)\frac{z}{2}} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \left\{\frac{2}{\left(\frac{f}{n_2}+\frac{1}{n_1}\right)} \mathrm{w}\right\}^{\frac{n_1+n_2}{2}-1} e^{-w} \frac{2}{\left(\frac{f}{n_2}+\frac{1}{n_1}\right)} dw$$

$$= \frac{f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \left\{\frac{2}{\left(\frac{f}{n_2}+\frac{1}{n_1}\right)}\right\}^{\frac{n_1+n_2}{2}} \mathrm{w}^{\frac{n_1+n_2}{2}-1} e^{-w} dz$$

$$= \frac{2^{\frac{n_1+n_2}{2}} \cdot f^{\frac{n_1}{2}-1}}{2^{\frac{n_1+n_2}{2}} n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^\infty \frac{1}{\left(\frac{f}{n_2}+\frac{1}{n_1}\right)^{\frac{n_1+n_2}{2}}} \mathrm{w}^{\frac{n_1+n_2}{2}-1} e^{-w} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}}{n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\left(\frac{\mathrm{f}}{n_2}+\frac{1}{n_1}\right)^{\frac{n_1+n_2}{2}}} \int_0^\infty \mathrm{w}^{\frac{n_1+n_2}{2}-1} e^{-w} dz$$

$$= \frac{f^{\frac{n_1}{2}-1}\Gamma\left(\frac{n_1+n_2}{2}\right)}{n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\left(\frac{\mathrm{f}}{n_2}+\frac{1}{n_1}\right)^{\frac{n_1+n_2}{2}}}$$

$$\because \int_0^\infty \mathrm{w}^{\frac{n_1+n_2}{2}-1} e^{-w} dz = \Gamma\left(\frac{n_1+n_2}{2}\right)$$

$$= \frac{f^{\frac{n_1}{2}-1}\Gamma\left(\frac{n_1+n_2}{2}\right)}{n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \left(\frac{f}{n_2}+\frac{1}{n_1}\right)^{\frac{n_1+n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{\mathrm{f}^{\frac{n_1}{2}-1}\Gamma\left(\frac{n_1+n_2}{2}\right)}{n_2^{\frac{n_1}{2}} n_1^{\frac{n_2}{2}} \frac{(n_1 f+n_2)^{\frac{n_1+n_2}{2}}}{(n_1 n_2)^{\frac{n_1+n_2}{2}}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} \cdot f^{\frac{n_1}{2}-1}\Gamma\left(\frac{n_1+n_2}{2}\right)}{(n_1 f+n_2)^{\frac{n_1+n_2}{2}} \Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)}$$

$$= \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}} \cdot f^{\frac{n_1}{2}-1}}{(n_1 f + n_2)^{\frac{n_1+n_2}{2}} \beta(n_1, n_2)}$$

$$\because \frac{\Gamma\left(\frac{n_1 + n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} = \frac{1}{\beta(n_1, n_2)}$$

$$P(f) = \frac{n_1^{\frac{n_1}{2}} n_2^{\frac{n_2}{2}}}{\beta(n_1, n_2)} \cdot \frac{f^{\frac{n_1}{2}-1}}{(n_1 f + n_2)^{\frac{n_1+n_2}{2}}}$$

Formula 25.